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Necessary and sufficient conditions for lossless negative imaginary systems☆

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Abstract

This paper studies some useful properties of lossless negative imaginary transfer matrices for both continuous-time and discrete-time systems. Necessary and sufficient conditions are established for lossless negative imaginary systems both in frequency domain and state-space realization. Meanwhile, a minor decomposition method for lossless negative imaginary systems is proposed in terms of a partial-fraction expansion. This method is important for developing non-proper lossless negative imaginary theory in this paper by allowing poles at the origin and infinity. Two new relationships between lossless positive real and lossless negative imaginary systems are consequently established. According to these new established relationships, a generalized continuous-time lossless negative imaginary lemma and a different version of discrete-time lossless negative imaginary lemma are developed in terms of a minimal state-space realization. Several examples are provided to illustrate the main results.

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1. Introduction

Positive realness, as known as passivity, is a notion used to describe a special class of energy-related systems with phase limited within $[-\frac{\pi}{2}, +\frac{\pi}{2}]$ [1–3]. The study of positive real control systems has obtained enormous successes both in theory and in practice in the past three decades [1,4]. However, one major limitation of positive real systems is that their relative degree must be zero or one [1]. The negative imaginary theory, which allows the relative degree of transfer matrix to be two and requires its phase in the interval $[-\pi, 0]$, has been developed as a useful complement to the passivity theory and the positive real theory [5]. Since the notion of negative imaginariness was first established in [6], many control theorists carried out an extensive study on negative imaginary systems in the past decade, ranging from robust stability analysis [7,8], generalization of negative imaginary concepts [9,10], state-space conditions [4], negative imaginary synthesis problem [3,11,12], networked negative imaginary systems [13], strictly negative imaginary systems [14] and the application on nanopositioning system [15]. More research results and applications on negative imaginary systems can be found in [16–21].

The concept of losslessness is also related to that of passivity [22]. An m-port network, assumed to be storing no energy at time t, is called lossless if it is passive and if, when a finite amount of energy is put into the network, all the energy can be extracted again [22]. It is well-known that systems which dissipate energy often result in positive real properties [1,22]. The so-called lossless positive real systems are those systems whose positive real transfer matrix $F(s)$ satisfies the condition: $F(j\omega) + F^*(j\omega) = 0$ for all real ω. That is, $F(j\omega) = -F^*(j\omega)$, the negative of a lossless positive real transfer matrix is its own complex conjugate transpose [23]. More physical descriptions about the lossless positive real systems can be found in [22–24]. Since the concept of lossless positive real systems first appeared in [23], the study of lossless positive real systems attracted much attention among control theorists [24–31]. For example, a matrix fraction description of lossless positive real property was gave in terms of a Hankel matrix [24]. The invariance of characteristic values and L_∞ norm under lossless positive real transformations were studied in [27,29]. The authors in [28] developed a lossless positive real lemma for descriptor systems. The continuous-time and discrete-time lossless positive real lemma were, respectively, developed in [22] and [30,31] in terms of state-space condition. Also, it is noted that lossless positive real transfer functions form a convex set, which admitted an important role in the proof of Kharitonov's Theorem (see [26]).

Lossless negative imaginary system is emerged as a special and important class of negative imaginary system whose transfer function matrix $G(s)$ satisfies lossless negative imaginary condition: $j[G(j\omega) - G^*(j\omega)] = 0$ for all $\omega > 0$ with $j\omega$ not a pole of $G(s)$. Such lossless negative imaginary properties have numerous applications in control of undamped flexible structures and lossless electrical circuits, see [22,23,32]. The definition of continuous-time proper lossless negative imaginary systems was first proposed in [32] by restricting no poles at the origin and infinity, and a minimal state-space characterization of such systems was also developed in [32]. Then, an algebraic approach to the realization of continuous-time lossless negative imaginary systems was studied in [33]. Subsequently, the work in [9] extended the definition of continuous-time lossless negative imaginary systems to non-proper case by allowing poles at the origin and infinity, and [34] first proposed the definition of discrete-time lossless negative imaginary systems.

However, article [34] did not systematic study the properties of the frequency-domain and state-space conditions on such lossless negative imaginary systems. Motivated by this and the numerous applications on such systems, we will further study some useful and important properties of lossless negative imaginary transfer matrices for both continuous-time and discrete-time systems in this paper. Via partial-fraction expansion, our results show that any continuous-time lossless negative imaginary transfer matrices can be decomposed as a sum of several lossless negative imaginary transfer matrices. Based on this minor decomposition property, two new relationships between lossless positive real and lossless negative imaginary transfer matrices are established for proper and non-proper cases. Then, a generalized lossless negative imaginary lemma in terms of minimal state-space condition is derived by allowing poles at the origin. Also, we extend all the results in continuous-time lossless negative imaginary systems to discrete-time systems. Moreover, a different version of the discrete-time lossless negative imaginary lemma is developed. The extended results of the paper show a nice parallel to the better understood results on non-proper lossless positive real systems [22,23].

The rest of the paper is organized as follows: Section 2 presents the definitions of lossless positive real and lossless negative imaginary transfer matrices for both continuous-time and discrete-time systems, and then we establish necessary and sufficient conditions in frequency domain for continuous-time lossless negative imaginary systems. Section 3 presents the minor decomposition theory of lossless negative imaginary transfer matrices in terms of the partial-fraction expansion. Section 4 establishes the new relationships between lossless positive real and lossless negative imaginary transfer matrices. A generalized lossless negative imaginary lemma in terms of state-space condition for continuous-time systems is developed in Section 5, and a different discrete-time lossless negative imaginary lemma in terms of state-space condition is also introduced in this section. Section 6 presents two examples to illustrate the lossless negative imaginary lemmas. Section 7 concludes the paper.

Notation: $\mathbb{R}^{m\times n}$ and $\mathcal{R}^{m\times n}$ denote the sets of $m\times n$ real matrices and real-rational proper transfer function matrices, respectively. Re[.] denotes the real part of complex numbers. A^T, $\bar{A}$ and A^* denote the transpose, the complex conjugate and the complex conjugate transpose of a complex matrix A, respectively. $A>0$ or $A\geq 0$ denotes a symmetric positive definite or symmetric positive semidefinite matrix. $G(s)\sim(A, B, C, D)$ or $G(z)\sim(A, B, C, D)$ denotes that (A, B, C, D) is a state-space realization of $G(s)$ or $G(z)$, where $G(s)$ and $G(z)$ are the transfer function matrix of a continuous-time system and a discrete-time system, respectively. I denotes any identity matrix with compatible dimensions.

2. Preliminaries and frequency domain conditions

In this section, we present the concepts and frequency domain conditions for lossless positive real, negative imaginary and lossless negative imaginary transfer matrices of continuous-time and discrete-time systems, and introduce some useful preliminary results.

2.1. Continuous-time systems

This subsection first recalls several important definitions of continuous-time lossless positive real, negative imaginary and lossless negative imaginary transfer matrices introduced in [9,22], and then presents the frequency domain conditions of continuous-time lossless negative imaginary systems. The definition of continuous-time positive real transfer matrices also can be found in [22]

Definition 1 [22]**.** A square real-rational transfer function matrix $F(s)$ is said to be lossless positive real if

1. $F(s)$ is positive real;
2. $F(j\omega) + F^*(j\omega) = 0$ for all $\omega > 0$ except values of ω where $j\omega$ is a pole of $G(s)$.

Definition 2 [9]**.** A square real-rational transfer function matrix $G(s)$ is said to be negative imaginary if

1. $G(s)$ has no poles in $\mathrm{Re}[s] > 0$;
2. $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega > 0$ except values of ω where $j\omega$ is a pole of $G(s)$;
3. if $s = 0$ is a pole of $G(s)$, then it is at most a double pole, $\lim_{s\to 0} s^2 G(s)$ is positive semidefinite Hermitian, and $\lim_{s\to 0} s^m G(s) = 0$ for all $m \geq 3$;
4. if $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, ω_0 is finite, it is at most a simple pole and the residue matrix $K = \lim_{s\to j\omega_0} (s - j\omega_0) jG(s)$ is positive semidefinite Hermitian;
5. if $s = j\infty$ is a pole of $G(s)$, then it is at most a double pole, $\lim_{\omega\to\infty} \frac{G(j\omega)}{(j\omega)^2}$ is negative semidefinite Hermitian, and $\lim_{\omega\to\infty} \frac{G(j\omega)}{(j\omega)^m} = 0$ for all $m \geq 3$.

Definition 3 [9]**.** A square real-rational transfer function matrix $G(s)$ is said to be lossless negative imaginary if

1. $G(s)$ is negative imaginary;
2. $j[G(j\omega) - G^*(j\omega)] = 0$ for all $\omega > 0$ except values of ω where $j\omega$ is a pole of $G(s)$.

The following theorem, which can be considered as a generalization of Lemma 2 in [32] by allowing poles at the origin and infinity, provides a necessary and sufficient condition in frequency domain for a system to be non-proper lossless negative imaginary.

Theorem 1. *A square real-rational transfer function matrix $G(s)$ is lossless negative imaginary if and only if*

1. *all poles of elements of $G(s)$ are purely imaginary;*
2. *if $s = 0$ is a pole of $G(s)$, it is at most a double pole, $\lim_{s\to 0} s^2 G(s)$ is positive semidefinite Hermitian, and $\lim_{s\to 0} s^m G(s) = 0$ for all $m \geq 3$;*
3. *if $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, ω_0 is finite, it is at most a simple pole and the residue matrix $K = \lim_{s\to j\omega_0} (s - j\omega_0) jG(s)$ is positive semidefinite Hermitian;*
4. *if $s = j\infty$ is a pole of $G(s)$, it is at most a double pole, $\lim_{\omega\to\infty} \frac{G(j\omega)}{(j\omega)^2}$ is negative semidefinite Hermitian, and $\lim_{\omega\to\infty} \frac{G(j\omega)}{(j\omega)^m} = 0$ for all $m \geq 3$;*
5. *$G(s) = G^T(-s)$ for all s such that s is not a pole of any element of $G(s)$.*

Proof. (Necessity) Suppose $G(s)$ is lossless negative imaginary. Condition 2 of Definition 3 implies that $j[G(j\omega) - G^*(j\omega)] = 0$ for all $\omega > 0$ except values of ω where $j\omega$ is a pole of $G(s)$. Then, we have $\overline{j[G(j\omega) - G^*(j\omega)]} = 0$ for all $\omega > 0$ with $j\omega$ not a pole of $G(s)$, that is, $j[G(j\omega) - G^*(j\omega)] = 0$ for all $\omega < 0$ with $j\omega$ not a pole of $G(s)$. According to the continuity of $G(s)$, it follows that $j[G(0) - G^*(0)] = 0$. Hence, we have $j[G(s) - G^T(-s)] = 0$ for all $s = j\omega$, where $j\omega$ is not a pole of $G(s)$. Because $j[G(s) - G^T(-s)]$ is an analytic function of s, it follows from maximum modulus theorem ([23, Theorem A4-3]) that $j[G(s) - G^T(-s)] = 0$ holds for all s such that s is not a pole of $G(s)$, and hence $G(s) = G^T(-s)$. Condition 5 holds.

Suppose s_0 is a pole of $G(s)$. It follows from Condition 5 that $-s_0$ is also a pole of $G(s)$. According to Definition 2, we know that $G(s)$ has no poles in $\mathrm{Re}[s] > 0$. If $\mathrm{Re}[s_0] < 0$, then $\mathrm{Re}[-s_0] > 0$, there exists contradiction. So, the only case is that all poles of elements of $G(s)$ lie on the imaginary axis. Condition 1 holds. Moreover, conditions 3–5 of Definition 2 imply that conditions 2–4 hold.

(Sufficiency) Suppose conditions 1–5 hold. Conditions 1–4 imply Condition 1 and conditions 3–5 of Definition 2 hold. Then, Condition 5 implies $j[G(j\omega) - G^*(j\omega)] = 0$ for all $\omega > 0$ such that $j\omega$ is not a pole of any element of $G(s)$. It follows from Definitions 2 and 3 that $G(s)$ is lossless negative imaginary. □

The following lemma extends the lossless positive real lemma in [22, page 229] by relaxing the observability requirement of (A, C) and the non-singularity requirement of P. Lemma 1 is useful in the proof of lossless negative imaginary lemma in Section 5. The detailed proof of Lemma 1 can be found in [35].

Lemma 1 [35,36]**.** *Let (A, B, C, D) be a state-space realization of a square real-rational proper transfer function matrix $F(s) \in \mathcal{R}^{m\times m}$, where (A, B) is completely controllable, (A, C) is not necessarily completely observable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and $m \le n$. Then, $F(s)$ is lossless positive real if and only if there exists a real matrix $P = P^T \ge 0$, $P \in \mathbb{R}^{n\times n}$, such that*

$$\begin{aligned} PA + A^T P &= 0, \\ PB - C^T &= 0, \\ D + D^T &= 0. \end{aligned} \tag{1}$$

The following lemma characterizes the properties of sum of any non-proper lossless negative imaginary transfer matrices.

Lemma 2.

1. *If $G_i(s)$, $i = 1, 2, \dots, n$, is lossless negative imaginary, then $\sum_{i=1}^{n} G_i(s)$ is lossless negative imaginary.*
2. *If $G_i(s)$, $i = 1, 2, \dots, n$, is lossless negative imaginary, and $G_j(s)$, $j = 1, 2, \dots, m$, is negative imaginary, then $\sum_{i=1}^{n} G_i(s) + \sum_{j=1}^{m} G_j(s)$ is negative imaginary.*

Proof. The proof is straightforward according to the definitions of negative imaginary and lossless negative imaginary transfer function matrices. □

2.2. Discrete-time systems

This subsection introduces the definitions and frequency domain conditions for discrete-time lossless positive real and lossless negative imaginary transfer matrices. The definition of discrete-time positive real and negative imaginary transfer matrices can be found in [37,38]

Definition 4 [26,39]**.** A square real-rational proper transfer function matrix $F(z)$ is said to be lossless positive real if

1. $F(z)$ is positive real;
2. $F(e^{j\theta}) + F^*(e^{j\theta}) = 0$ for all $\theta \in (0, \pi)$ except values of θ where $e^{j\theta}$ is a pole of $F(z)$.

Definition 5 [34]**.** A square real-rational proper transfer function matrix $G(z)$ is said to be lossless negative imaginary if

1. $G(z)$ is negative imaginary;
2. $j[G(e^{j\theta}) - G^{*}(e^{j\theta})] = 0$ for all $\theta \in (0, \pi)$ except values of θ where $e^{j\theta}$ is a pole of $G(z)$.

The following lemma provides a necessary and sufficient condition for a system to be discrete-time lossless negative imaginary in terms of frequency domain conditions. The detail proof of Lemma 3 can be found in [34].

Lemma 3 [34]**.** *A square real-rational proper transfer function matrix $G(z)$ is lossless negative imaginary if and only if*

1. *all poles of elements of $G(z)$ lie on $|z| = 1$;*
2. *if $z_0 = e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, is a pole of $G(z)$, then it is a simple pole and the corresponding residue matrix $\tilde{K} = \lim_{z\to z_0}(z - z_0)jG(z)$ at any pole $z_0 = e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, satisfies that $e^{-j\theta_0}\tilde{K}$ is positive semidefinite Hermitian;*
3. *if $z = 1$ is a pole of $G(z)$, then $\lim_{z\to 1}(z - 1)^2 G(z)$ is positive semidefinite Hermitian, and $\lim_{z\to 1}(z - 1)^m G(z)=0$ for all $m \geq 3$;*
4. *if $z = -1$ is a pole of $G(z)$, then $\lim_{z\to -1}(z + 1)^2 G(z)$ is negative semidefinite Hermitian, and $\lim_{z\to -1}(z + 1)^m G(z) = 0$ for all $m \geq 3$;*
5. *$G(z) = G^T(z^{-1})$ for all z such that z is not a pole of $G(z)$.*

The following lemma is the discrete-time KYP lemma for lossless system derived in [31], which can be considered as the discrete-time lossless positive real lemma.

Lemma 4 [31]**.** *Let (A, B, C, D) be a minimal state-space realization of $F(z) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and $m \leq n$. Then, necessary and sufficient conditions for $F(z)$ to be lossless positive real are that there exists a real matrix $P = P^T > 0$, $P \in \mathcal{R}^{n\times n}$, such that*

$$\begin{aligned} A^T PA - P &= 0, \\ B^T PA - C &= 0, \\ D + D^T - B^T PB &= 0. \end{aligned}$$

The following lemma characterizes the sum properties of any discrete-time lossless negative imaginary transfer matrices, which can be considered as the discrete-time counterpart of Lemma 2. The proof is trivial.

Lemma 5.

1. *If $G_i(z)$, $i = 1, 2, \ldots, n$, is lossless negative imaginary, then $\sum_{i=1}^{n} G_i(z)$ is lossless negative imaginary.*
2. *If $G_i(z)$, $i = 1, 2, \ldots, n$, is lossless negative imaginary, and $G_j(z)$, $j = 1, 2, \ldots, m$, is negative imaginary, then $\sum_{i=1}^{n} G_i(z) + \sum_{j=1}^{m} G_j(z)$ is negative imaginary.*

3. Partial-fraction expansion of lossless negative imaginary transfer matrices

In this section, we consider the minor decomposition theory of lossless negative imaginary properties in terms of a partial-fraction expansion for both continuous-time and discrete-time systems, which provide the **core** to develop the lossless negative imaginary theory in this paper.

3.1. Continuous-time systems

This subsection studies the partial-fraction expansion of continuous-time lossless negative imaginary systems. Suppose that $G(s)$ is a square real-rational lossless negative imaginary transfer matrix. Then, define the following four matrices:

$$\begin{aligned} &A_2 = \lim_{\omega\to\infty} \frac{G(j\omega)}{(j\omega)^2}, \quad A_1 = \lim_{\omega\to\infty} \frac{(G(j\omega) - (j\omega)^2 A_2)}{j\omega}, \\ &C_2 = \lim_{s\to 0} s^2 G(s), \qquad C_1 = \lim_{s\to 0} s\left(G(s) - \frac{C_2}{s^2}\right). \end{aligned} \tag{2}$$

According to Theorem 1, it follows that $A_2 = A_2^* \le 0$ and $C_2 = C_2^* \ge 0$. Note that $j\omega$ is a pole of $G(s)$, the $-j\omega$ must also be a pole of $G(s)$, that is, $j\omega$ and $-j\omega$ occur in pairs. Because all the poles of lossless negative imaginary transfer matrices lie on the imaginary axis, $G(s)$ can be decomposed by using the residue matrix properties about poles on the imaginary axis as the following form:

$$\begin{aligned} G(s) &= \sum_i \frac{-jK_i}{s - j\omega_i} + \sum_i \frac{jK^*}{s + j\omega_i} + \frac{1}{s}C_1 + \frac{1}{s^2}C_2 + sA_1 + s^2A_2 + G(\infty) \\ &= \sum_i \frac{sQ_i + T_i}{s^2 + \omega_i^2} + \frac{1}{s}C_1 + \frac{1}{s^2}C_2 + sA_1 + s^2A_2 + G(\infty), \end{aligned}$$

where K_i is the residue matrix of $jG(s)$ at $j\omega_i$, $A_2 = A_2^* \le 0$, $C_2 = C_2^* \ge 0$, $Q_i = j(K_i^* - K_i)$, and $T_i = \omega_i(K_i + K_i^*)$. According to Condition 3 of Theorem 1, we know $K_i = K_i^* \ge 0$, it follows that $Q_i = 0$ and $T_i = T_i^*$. Then, we have

$$G(s) = \sum_i \frac{T_i}{s^2 + \omega_i^2} + \frac{1}{s}C_1 + \frac{1}{s^2}C_2 + sA_1 + s^2A_2 + G(\infty). \tag{3}$$

We can find that $\sum_i \frac{T_i}{s^2+\omega_i^2}$, $\frac{1}{s^2}C_2$ and s^2A_2 are lossless negative imaginary. The fact that they are negative imaginary is immediate from the definition of negative imaginary systems, and the lossless property follows by observing that

$$\begin{aligned} &j\left[\frac{1}{(j\omega)^2}C_2 - \frac{1}{(-j\omega)^2}C_2^*\right] = 0; \\ &j[(j\omega)^2A_2 - (-j\omega)^2A_2^*] = 0; \\ &j\left[\sum_i \frac{T_i}{(j\omega)^2 + \omega_i^2} - \sum_i \frac{T_i^*}{(-j\omega)^2 + \omega_i^2}\right] = 0. \end{aligned}$$

Next, we study the properties of matrices A_1 and C_1 in the following lemma, where A_1 and C_1 are defined in Eq. (2).

Lemma 6. *Given a square real-rational lossless negative imaginary transfer function matrix $G(s)$. Then $A_1 + A_1^* = 0$ and $C_1 + C_1^* = 0$ hold.*

Proof. Suppose $G(s)$ is lossless negative imaginary. It follows that $G(s)$ has at most a double pole at infinity and the origin. First, we will prove $A_1 + A_1^* = 0$. When $G(s)$ has no poles at infinity, one has that $A_2 = 0$, $A_1 = 0$, and hence $A_1 + A_1^* = 0$. When $G(s)$ has a simple pole at infinity, $A_2 = 0$, let

$$G(s) = G_0(s) + sA_1 + G(\infty),$$

where $G_0(s)$ is strictly proper, and $G(\infty) = G^T(\infty)$. $G(s)$ and $G_0(s)$ have the same poles except at infinity. Condition 2 of Definition 3 implies that

$$\begin{aligned} j\big[G(j\omega) - G^*(j\omega)\big] &= j\big[j\omega A_1 + G_0(j\omega) + G(\infty) + j\omega A_1^* - G_0^*(j\omega) - G^*(\infty)\big] \\ &= -\omega\big(A_1 + A_1^*\big) + j\big[G_0(j\omega) - G_0^*(j\omega)\big] = 0, \end{aligned} \tag{4}$$

for all $\omega > 0$, where $j\omega$ is not a pole of $G(s)$ and $G_0(s)$. Suppose $A_1 + A_1^* \neq 0$. Then, since $G_0(s)$ is strictly proper, there exists a sufficiently large ω_1 such that $j[G_0(j\omega_1) - G_0^*(j\omega_1)] = 0$, which contradicts with Eq. (4). So, $A_1 + A_1^* = 0$.

When $G(s)$ has a double pole at infinity, let

$$G(s) = G_0(s) + sA_1 + s^2A_2 + G(\infty),$$

where $A_2 = A_2^* \leq 0$, $G(\infty) = G^T(\infty)$ and $G_0(s)$ is strictly proper. Similarly, Condition 2 of Definition 3 implies that $j[G(j\omega) - G^*(j\omega)] = -\omega(A_1 + A_1^*) + j[G_0(j\omega) - G_0^*(j\omega)] = 0$. Using the similar analysis as the case where $G(s)$ has a simple pole at infinity, we have $A_1 + A_1^* = 0$.

Next, we will prove $C_1 + C_1^* = 0$. When $G(s)$ has no poles at the origin, one has that $C_2 = 0$, $C_1 = 0$, and hence $C_1 + C_1^* = 0$. When $G(s)$ has a simple pole at the origin, $C_2 = 0$, let

$$G(s) = \sum_i \frac{T_i}{s^2 + \omega_i^2} + \frac{1}{s}C_1 + sA_1 + s^2A_2 + G(\infty),$$

where $T_i = T_i^*$, $G(\infty) = G^T(\infty)$, $A_1 + A_1^* = 0$ and $A_2 = A_2^* \leq 0$. Condition 2 of Definition 3 implies that

$$\begin{aligned} j[G(j\omega) - G^*(j\omega)] =& j\Bigg[\sum_i \frac{T_i}{(j\omega)^2 + \omega_i^2} + \frac{1}{j\omega}C_1 + (j\omega)A_1 + (j\omega)^2A_2 + G(\infty) \\ & - \sum_i \frac{T_i^*}{(-j\omega^2) + \omega_i^2} - \frac{1}{(-j\omega)}C_1 - (-j\omega)A_1 - (-j\omega)^2A_2 - G^*(\infty)\Bigg] \\ =& -\omega(A_1 + A_1^*) + \frac{1}{\omega}(C_1 + C_1^*)] = 0, \end{aligned}$$

for all $\omega > 0$, where $j\omega$ is not a pole of $G(s)$. Because $-\omega(A_1 + A_1^*) = 0$, it follows that $C_1 + C_1^* = 0$.

When $G(s)$ has a double pole at the origin, let

$$G(s) = \sum_i \frac{T_i}{s^2 + \omega_i^2} + \frac{1}{s}C_1 + \frac{1}{s^2}C_2 + sA_1 + s^2A_2 + G(\infty),$$

where $T_i = T_i^*$, $G(\infty) = G^T(\infty)$, $A_1 + A_1^* = 0$, $A_2 = A_2^* \leq 0$ and $C_2 = C_2^* \geq 0$. Then, Condition 2 of Definition 3 with $A_1 + A_1^* = 0$ implies that

$$j\big[G(j\omega) - G^*(j\omega)\big] = \frac{1}{\omega}(C_1 + C_1^*)] = 0.$$

This completes the proof. □

Remark 1. Let any lossless negative imaginary transfer matrix be the form of Eq. (3). It can be found that $j[G(j\omega) - G^*(j\omega)] = \frac{-\omega^2(A_1+A_1^*)+(C_1+C_1^*)}{\omega} = 0$ for all $\omega > 0$ except values of ω where $j\omega$ is a pole of $G(s)$. It follows from Lemma 2 in [9] and Lemma 3 in [40], we know that $A_1 + A_1^* \leq 0$ and $C_1 + C_1^* \geq 0$. So, the only case is that $A_1 + A_1^* = 0$ and $C_1 + C_1^* = 0$ for lossless negative imaginary systems. More details are given in the proof of Lemma 6 to show the different cases which could appear in lossless negative imaginary transfer matrices. If $G(s)$ is a symmetric lossless negative imaginary transfer matrix, it follows that $A_1 = A_1^* \leq 0$ and $C_1 = C_1^* \geq 0$, and hence, $A_1 = 0$ and $C_1 = 0$. In other words, it is impossible for symmetric lossless negative imaginary transfer matrix having simple pole at the origin and infinity.

The following theorem gives a decomposed property about lossless negative imaginary transfer matrices.

Theorem 2. *Let $G(s)$ be a square real-rational transfer function matrix of the form*

$$G(s) = G_0(s) + \frac{1}{s}C_1 + \frac{1}{s^2}C_2 + sA_1 + s^2A_2 + G(\infty),$$

where $G_0(s)$ is strictly proper, and $G_0(s)$ has no poles at the origin and infinity. Then, $G(s)$ is lossless negative imaginary if and only if $G_0(s)$ is lossless negative imaginary, $A_2 = A_2^* \leq 0$, $C_2 = C_2^* \geq 0$, $A_1 + A_1^* = 0$, $C_1 + C_1^* = 0$ and $G(\infty) = G^T(\infty)$.

Proof. (Necessity) Suppose that $G(s)$ is lossless negative imaginary. According to Lemmas 1 and 6, it follows that $A_2 = A_2^* \leq 0$, $C_2 = C_2^* \geq 0$, $A_1 + A_1^* = 0$, $C_1 + C_1^* = 0$ and $G(\infty) = G^T(\infty)$. $G(s)$ and $G_0(s)$ have the same poles except at the origin and infinity. For $\omega > 0$, $j\omega$ is not a pole of $G(s)$ and $G_0(s)$, we have $j[G(j\omega) - G^*(j\omega)] = j[G_0(j\omega) - G_0^*(j\omega)] = 0$. If $j\omega_0$, $\omega_0 > 0$ is a pole of $G(s)$, then the residue matrix $\lim_{s\to j\omega_0}(s - j\omega_0)jG(s) = \lim_{s\to j\omega_0}(s - j\omega_0)jG_0(s)$, is positive semidefinite. Hence, $G_0(s)$ is lossless negative imaginary.

(Sufficiency) Suppose that $G_0(s)$ is lossless negative imaginary, $A_2 = A_2^* \leq 0$, $C_2 = C_2^* \geq 0$, $A_1 + A_1^* = 0$, $C_1 + C_1^* = 0$ and $G(\infty) = G^T(\infty)$. It follows that $\frac{1}{s}C_1$, $\frac{1}{s^2}C_2$, sA_1, s^2A_2 are lossless negative imaginary. Then, according to the sum properties of lossless negative imaginary systems in Lemma 2, $G(s)$ is lossless negative imaginary. □

Remark 2. According to the analysis in this subsection, we can notice that any lossless negative imaginary transfer matrices can be regarded as a sum of several lossless negative imaginary transfer matrices. For example, consider the transfer function $G(s) = \frac{1-2s^4}{s^2(s^2+1)}$ in [9, Example 8]. $G(s)$ can be decomposed as $G(s) = \frac{1}{s^2} + \frac{1}{s^2+1} - 2$, where $C_2 = 1$, $G(\infty) = -2$, the residue matrix of $jG(s)$ at $s = j$ is $K = \frac{1}{2}$, and hence $T_1 = K + K^* = 1$. Both $\frac{1}{s^2}$ and $\frac{1}{s^2+1}$ are lossless negative imaginary. Moreover, the negative imaginary transfer matrices also have similar properties as Theorem 2. We are able to decompose any negative imaginary transfer matrix $G(s)$ into the sum of a lossless negative imaginary transfer matrix and a negative imaginary transfer matrix. If $G(s)$ is symmetric, we have $G(s)$ is negative imaginary if and only if $A_2 = A_2^* \leq 0$, $C_2 = C_2^* \geq 0$, $A_1 = A_1^* \leq 0$, $C_1 = C_1^* \geq 0$, $G(\infty) = G^T(\infty)$ and $G_0(s)$ is negative imaginary.

Decompose the lossless negative imaginary transfer matrices into a proper part and a non-proper part. We can directly have the following result.

Corollary 1. *Let $G(s)$ be a square real-rational transfer function matrix of the form*

$$G(s) = G_0(s) + sA_1 + s^2A_2 + \sum_{i=3}^{m} s^i A_i,$$

where $G_0(s)$ has no poles at infinity. Then, $G(s)$ is lossless negative imaginary if and only if $G_0(s)$ is lossless negative imaginary, $A_2 = A_2^* \leq 0$, $A_1 + A_1^* = 0$ and $A_i = 0$ for $i \geq 3$.

Remark 3. Corollary 1 is useful in deriving the non-proper descriptor lossless negative imaginary lemma. Also, for the non-proper negative imaginary transfer matrices, we have a similar result, that is, $G(s)$ is negative imaginary if and only if $A_2 = A_2^* \leq 0$ and $G_0(s) + sA_1$ is negative imaginary, which is also useful in the study of non-proper descriptor negative imaginary systems. If $G(s)$ is symmetric, we have $G(s)$ is negative imaginary if and only if $A_2 = A_2^* \leq 0$, $A_1 = A_1^* \leq 0$, $A_i = 0$ for $i \geq 3$ and $G_0(s)$ is negative imaginary.

3.2. Discrete-time systems

This subsection studies the partial-fraction expansion of discrete-time lossless negative imaginary transfer matrices. Suppose that $G(z)$ is a lossless negative imaginary transfer matrix. Define the following four matrices,

$$A_2 = \lim_{z\to 1}(z-1)^2 G(z), \quad A_1 = \lim_{z\to 1}(z-1)\left(G(z) - \frac{A_2}{(z-1)^2}\right),$$
$$C_2 = \lim_{z\to -1}(z+1)^2 G(z), \quad C_1 = \lim_{z\to -1}(z+1)\left(G(z) - \frac{C_2}{(z+1)^2}\right).$$

We have following properties about poles at $z = \pm 1$.

Lemma 7. *Given a square real-rational proper lossless negative imaginary transfer function matrix $G(z)$. We have:*

1. *If $G(z)$ has a double pole at $z = 1$, then $A_1 + A_1^* = A_2 + A_2^* \geq 0$;*
2. *If $G(z)$ has a simple pole at $z = 1$, then $A_1 + A_1^* = 0$;*
3. *If $G(z)$ has a double pole at $z = -1$, then $C_1 + C_1^* = -(C_2 + C_2^*) \geq 0$;*
4. *If $G(z)$ has a simple pole at $z = -1$, then $C_1 + C_1^* = 0$.*

Proof. Because $G(z)$ is discrete-time lossless negative imaginary system, $G(z)$ has at most a double pole at $z = \pm 1$. When $G(z)$ has no poles at $z = \pm 1$, it follows that $A_2 = A_1 = 0$, $C_2 = C_1 = 0$.

- Proof of Item 1: When $G(z)$ has a double pole at $z = 1$, similar to continuous-time case, we can write $G(z)$ in the form

$$G(z) = G_1(z) + \frac{A_1}{z-1} + \frac{A_2}{(z-1)^2}, \tag{5}$$

where $G_1(z)$ has no poles at $z = 1$ and $A_2 = A_2^* \geq 0$. By means of the bilinear transformation

$$s = \frac{z-1}{z+1}, \ \ z = \frac{1+s}{1-s}, \tag{6}$$

Eq. (5) is transformed into

$$\begin{aligned} G(s) &= G_1\left(\frac{1+s}{1-s}\right) + \frac{A_1}{\frac{1+s}{1-s} - 1} + \frac{A_2}{(\frac{1+s}{1-s} - 1)^2} \\ &= G_0(s) + \frac{A_1 - A_2}{2s} + \frac{A_2}{4s^2}, \end{aligned}$$

where $G_0(s) = G_1(\frac{1+s}{1-s}) - \frac{A_1}{2} + \frac{A_2}{4}$ has no poles at $s = 0$. According to Lemma 6 in [34], it follows that $G(s)$ is continuous-time lossless negative imaginary, and $G(s)$ has a double pole at $s = 0$. Then, according to Lemma 6, it follows that $\frac{A_1 - A_2}{2} + (\frac{A_1 - A_2}{2})^* = 0$. Hence, we have

$$A_1 + A_1^* = A_2 + A_2^* \geq 0.$$

- Proof of Item 2: When $G(z)$ has a simple pole at $z = 1$, let

$$G(z) = G_1(z) + \frac{A_1}{z-1}, \tag{7}$$

where $G_1(z)$ has no poles at $z = 1$. Using the bilinear transformation in (6), Eq. (7) is transformed into

$$G(s) = G_1\left(\frac{1+s}{1-s}\right) - \frac{1}{2}A_1 + \frac{1}{2s}A_1.$$

According to [34, Lemma 6], it follows that $G(s)$ is continuous-time lossless negative imaginary, and $G(s)$ has a simple pole at $s = 0$. Then, according to Lemma 6, it follows that $\frac{1}{2}A_1 + \frac{1}{2}A_1^* = 0$. Hence, we have $A_1 + A_1^* = 0$.

- Proof of Item 3: When $G(z)$ has a double pole at $z = -1$, let

$$G(z) = G_1(z) + \frac{C_1}{z+1} + \frac{C_2}{(z+1)^2}, \tag{8}$$

where $G_1(z)$ has no poles at $z = -1$ and $C_2 = C_2^* \leq 0$. Using the bilinear transformation in Eq. (6), Eq. (8) is transformed into

$$\begin{aligned} G(s) &= G_1\left(\frac{1+s}{1-s}\right) + \frac{C_1}{\frac{1+s}{1-s} + 1} + \frac{C_2}{(\frac{1+s}{1-s} + 1)^2} \\ &= G_0(s) + \frac{-C_1 - C_2}{2}s + \frac{C_2}{4}s^2, \end{aligned}$$

where $G_0(s) = G_1(\frac{1+s}{1-s}) + \frac{C_1}{2} + \frac{C_2}{4}$ has no poles at $s = \infty$. According to Lemma 6, it follows that

$$\left(-\frac{C_1 + C_2}{2}\right) + \left(-\frac{C_1 + C_2}{2}\right)^* = 0,$$

that is,

$$C_1 + C_1^* = -(C_2 + C_2^*) \geq 0.$$

- Proof of Item 4: When $G(z)$ has a simple pole at $z = -1$, let

$$G(z) = G_1(z) + \frac{C_1}{z+1}, \tag{9}$$

where $G_1(z)$ has no poles at $z = -1$. Using the bilinear transformation in (6), Eq. (9) is transformed into

$$G(s) = G_1(\frac{1+s}{1-s}) + \frac{C_1}{2} + \frac{-C_1}{2}s.$$

According to Lemma 6, it follows that $-\frac{1}{2}C_1 - \frac{1}{2}C_1^* = 0$. Hence, we have $C_1 + C_1^* = 0$.
□

Remark 4. It can be seen that Condition 2 of Lemma 7 is a special case of Condition 1 of Lemma 7 with $A_2 = 0$. Also, Condition 4 of Lemma 7 is a special case of Condition 3 of Lemma 7 with $C_2 = 0$. Suppose that $G(z)$ is a symmetric transfer function matrix. If $G(z)$ has a simple pole at $z = 1$, then $A_1 = 0$. If $G(z)$ has a simple pole at $z = -1$, then $C_1 = 0$. In other words, it is impossible for a symmetric lossless negative imaginary transfer function matrix to have simple poles at $z = \pm 1$. If $G(z)$ has a double pole at $z = 1$, then $A_1 = A_2 \geq 0$. If $G(z)$ has a double pole at $z = -1$, then $C_1 = -C_2 \geq 0$. Those facts follow from Lemma 6 and Remark 1. For example, consider two lossless negative imaginary transfer functions $G_1(z) = \frac{(z+1)^2}{(z-1)^2}$ and $G_2(z) = \frac{-(z-1)^2}{(z+1)^2}$. $G_1(z)$ has a double pole at $z = 1$, and $G_2(z)$ has a double pole at $z = -1$. We have $A_2 = \lim_{z\to 1}(z-1)^2 G_1(z) = 4$, $G_1(\infty) = 1$, $C_2 = \lim_{z\to -1}(z+1)^2 G_2(z) = -4$, and $G_2(\infty) = -1$. Then, $A_1 = A_2 = 4$, $G_1(z)$ can be decomposed as $G_1(z) = \frac{4}{(z-1)^2} + \frac{4}{z-1} + 1$, and $C_1 = -C_2 = 4$, $G_2(z)$ can be decomposed as $G_2(z) = \frac{-4}{(z+1)^2} + \frac{4}{z-1} - 1$.

4. Relationship between lossless positive real and lossless negative imaginary transfer matrices

4.1. Continuous-time systems

In this subsection, two new relationships between continuous-time lossless negative imaginary and lossless positive real transfer matrices will be, respectively, established in the non-proper and the proper cases. First, we present a new description of the relationship between the non-proper lossless negative imaginary and non-proper lossless positive real transfer matrices.

Lemma 8. *Let $G(s)$ be a square real-rational transfer function matrix. Supposing $G(s)$ has no poles at the origin, then $G(s)$ is lossless negative imaginary if and only if*

1. $G(0) = G^T(0)$;
2. $F(s) = -\frac{1}{s}[G(s) - G(0)]$ *is lossless positive real.*

Proof. (Necessity) Suppose that $G(s)$ is lossless negative imaginary. It follows from [9, Lemma 9] that $G(0) = G^T(0)$. When $G(s)$ has no pole at infinity, $F(s)$ has no poles at infinity. When $G(s)$ has a simple pole at infinity, then $F(s)$ has also no poles at infinity. Let $G(s) = sA_1 + G_0(s)$, where $G_0(s)$ is proper and $A_1 + A_1^T = 0$. Then, $F(s) = -A_1 - \frac{1}{s}G_0(s) + \frac{1}{s}G(0)$. As $\omega \to \infty$, it follows that $F(j\omega) + F^*(j\omega) = -(A_1 + A_1^T) = 0$. When $G(s)$ has a double pole at infinity, then $F(s)$ has a simple pole at infinity. The rest of the proof is the same as the necessity proof of [9, Lemma 9].

(Sufficiency) The sufficient proof is the same as the sufficient proof of [9, Lemma 9]. □

Example 1. To illustrate the usefulness of Lemma 8, consider a non-proper transfer matrix $G(s) = \begin{pmatrix} \frac{-s^2}{s^2+1} & -s \\ s & \frac{-s^2}{s^2+1} \end{pmatrix}$. $G(s)$ has no poles in $\mathrm{Re}[s] > 0$. A calculation shows that $j[G(j\omega) - G^*(j\omega)] = 0$. $G(s)$ has a simple pole at infinity and $s = j$. The residue matrix of $jG(s)$ at $s = j$ is positive semidefinite Hermitian, being $K = \lim_{s\to j}(s - j)jG(s) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Moreover, $\lim_{\omega\to\infty} \frac{G(j\omega)}{(j\omega)^2} = 0$ and $A_1 = \lim_{\omega\to\infty} \frac{G(j\omega)}{j\omega} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which satisfies $A_1 + A_1^* = 0$. According to Definitions 2 and 3, it follows that $G(s)$ is lossless negative imaginary. We can say that $G(s)$ is lossless negative imaginary if and only if $F(s) = -\frac{1}{s}[G(s) - G(0)] = \begin{pmatrix} \frac{s}{s^2+1} & 1 \\ -1 & \frac{s}{s^2+1} \end{pmatrix}$ is lossless positive real and $G(0) = G^T(0)$. A calculation shows that $G(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $F(s)$ satisfies all conditions in Definition 1: $F(s)$ is positive real, and $F(j\omega) + F^*(j\omega) = 0$ for all ω with $j\omega$ not a pole of $F(s)$.

When $G(s)$ is a real-rational proper transfer matrix, we have the following result:

Lemma 9. *Let $G(s)$ be a square real-rational proper transfer function matrix. Then $G(s)$ is lossless negative imaginary if and only if*

1. $G(\infty) = G^T(\infty)$;
2. $F(s) = s[G(s) - G(\infty)]$ *is lossless positive real.*

Proof. (Necessity) Suppose that $G(s)$ is lossless negative imaginary. It follows from [9, Lemma 11] that $G(\infty) = G^T(\infty)$. When $G(s)$ has no pole at the origin, $F(s)$ has no poles at the origin, and $F(0) + F^*(0) = 0$. When $G(s)$ has a simple pole at the origin, then $F(s)$ has also no poles at the origin. Let $G(s) = \frac{1}{s}C_1 + G_0(s)$, where $G_0(s)$ has no poles at the origin and $C_1 + C_1^* = 0$. Then, $F(s) = C_1 + sG_0(s) - sG(\infty)$, and hence, $F(0) + F^*(0) = C_1 + C_1^T = 0$. When $G(s)$ has a double pole at the origin, then $F(s)$ has a simple pole at the origin. The rest of the proof is the same as the necessity proof of [9, Lemma 11].

(Sufficiency) The sufficient proof is the same as the sufficient proof of [9, Lemma 11]. □

Example 2. As an illustration of Lemma 9, consider a proper transfer matrix $G(s) = \begin{pmatrix} \frac{1}{s^2+1} & \frac{1}{s} + 1 \\ \frac{-1}{s} + 1 & \frac{1}{s^2+1} \end{pmatrix}$. It can be found that $G(s)$ is lossless negative imaginary if and only if $F(s) = s[G(s) - G(\infty)] = \begin{pmatrix} \frac{s}{s^2+1} & 1 \\ -1 & \frac{s}{s^2+1} \end{pmatrix}$ is lossless positive real and $G(\infty) = G^T(\infty)$. A calculation shows that $G(\infty) = G^T(\infty) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Meanwhile, $G(s)$ and $F(s)$ satisfy all conditions in Definition 3 and Definition 1, respectively. Note that $G(s)$ has a simple pole at the origin, $C_1 = \lim_{s\to 0} sG(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ satisfies $C_1 + C_1^T = 0$, and $F(s)$ has no poles at the origin.

Remark 5. Lemma 8 can be considered as a generalization of Lemma 9 in [9] by allowing simple pole at infinity. Lemma 9 can be considered as a generalization of Lemma 11 in [9] by allowing simple pole at the origin. When $G(s)$ is a symmetric transfer matrix, Condition 1 in Lemmas 8 and 9 are redundant. The reason is that Condition 1 in Lemmas 8 and 9 are obvious for symmetric transfer matrices.

4.2. Discrete-time systems

In this subsection, two relationships between the discrete-time lossless negative imaginary and lossless positive real transfer matrices will be established under different assumptions. First, we present the relationship under the assumption that $G(z)$ has no poles at $z=-1$.

Lemma 10. *Let $G(z)$ be a square real-rational proper transfer function matrix, and supposing that $G(z)$ has no poles at $z=-1$. Then $G(z)$ is lossless negative imaginary if and only if*

1. $G(-1)=G(-1)^T$;
2. $F(z)=\frac{z-1}{z+1}(G(z)-G(-1))$ *is lossless positive real.*

Proof. (Necessity) Suppose $G(z)$ is lossless negative imaginary. Then, $G(z)$ is negative imaginary. It follows from Lemma 9 in [38] that $F(z)=\frac{z-1}{z+1}(G(z)-G(-1))$ is positive real and $G(-1)=G(-1)^T$. $G(z)$ and $F(z)$ have the same poles except at $z=1$. For any $e^{j\theta}$, $\theta\in(0,\pi)$, is not a pole of $G(z)$ and $F(z)$, Condition 2 of Definition 5 implies that

$$F(e^{j\theta})+F^*(e^{j\theta})=j\frac{\sin\theta}{1+\cos\theta}\left[G(e^{j\theta})-G^*(e^{j\theta})\right]=0.$$

If $G(z)$ has no poles at $z=1$ ($\theta=0$), then $F(z)$ has no poles at $z=1$ and $F(1)$=0. As a result, $F(1)+F^T(1)=0$. If $G(z)$ has a simple pole at $z=1$, then $F(z)$ has no poles at $z=1$. Let $G(z)=\frac{A_1}{z-1}+G_1(z)$, where $A_1+A_1^T=0$ and $G_1(z)$ is analytic in $|z|>1$ and at $z=\pm1$. Then,

$$F(z)=\frac{A_1}{z+1}+\frac{z-1}{z+1}G_1(z)-\frac{z-1}{z+1}G(-1).$$

It follows that

$$F(1)+F^T(1)=\frac{A_1+A_1^T}{2}=0.$$

Therefore, $F(e^{j\theta})+F^*(e^{j\theta})=0$ for all $\theta\in[0,\pi)$ with $e^{j\theta}$ not a pole of $F(z)$. A complex conjugate implies that $\overline{F(e^{j\theta})+F^*(e^{j\theta})}=0$ for all $\theta\in[0,\pi)$. That is, $F(e^{j\theta})+F^*(e^{j\theta})=0$ for all $\theta\in(-\pi,0]$ with $e^{j\theta}$ not a pole of $F(z)$. Note that $G(z)-G(-1)$ has a blocking zero at $z=-1$. So $F(z)$ has no poles at $z=-1$ ($\theta=\pm\pi$), and $F(-1)+F^*(-1)=0$ in view of the continuity of $F(z)$. Furthermore, $\theta\in[0,\pi]\cup[-\pi,0]$ is equal to $\theta\in[0,2\pi]$. Hence, it follows that

$$F(e^{j\theta})+F^*(e^{j\theta})=0,$$

for all $\theta\in[0,2\pi]$ with $e^{j\theta}$ not a pole of $F(z)$. If $G(z)$ has a double pole at $z=1$, then $F(z)$ has a simple pole at $z=1$. According to Definition 4, $F(z)=\frac{z-1}{z+1}(G(z)-G(-1))$ is lossless positive real.

(Sufficiency) Suppose $F(z)=\frac{z-1}{z+1}(G(z)-G(-1))$ is lossless positive real and $G(-1)=G^T(-1)$. $F(z)$ is also positive real. Then, it follows from [38, Lemma 9] that $G(z)=$

$\frac{z+1}{z-1}F(z)+G(-1)$ is negative imaginary, and $j[G(-1)-G^*(-1)]=0$. Condition 2 of Definition 4 implies that

$$j\left[G(e^{j\theta})-G^*(e^{j\theta})\right]=\frac{\sin\theta}{1-\cos\theta}\left[F(e^{j\theta})+F^*(e^{j\theta})\right]=0,$$

for all $\theta\in(0,\ \pi)$ with $e^{j\theta}$ not a pole of $G(z)$. If $F(z)$ has no poles at $z=1$ and $F(1)=0$, then $G(z)$ has no poles at $z=1$ and $j[G(1)-G^*(1)]=0$. If $F(1)\neq 0$ or $F(z)$ has a simple pole at $z=1$, then $G(z)$ has a simple pole or a double pole at $z=1$. Hence, according to Definition 5, $G(z)$ is lossless negative imaginary. □

Example 3. As an illustration of Lemma 10, consider the following transfer matrix $G(z)=\begin{pmatrix}\frac{(z+1)^2}{2(z^2+1)} & \frac{1+z}{2(1-z)}\\ \frac{z+1}{2(z-1)} & \frac{(z+1)^2}{2(z^2+1)}\end{pmatrix}$. $j[G(e^{j\theta}-G^*(e^{j\theta})]=0$ for all $\theta\in(0,\ \pi)$ with $e^{j\theta}$ not a pole of $G(z)$. $G(z)$ has simple pole at $z=e^{j\frac{\pi}{2}}=j$ and $z=1$. The residue matrix of $jG(z)$ at $z=j$ is given by $K=\begin{pmatrix}\frac{1}{2}j & 0\\ 0 & \frac{1}{2}j\end{pmatrix}$, and the matrix $e^{-j\frac{\pi}{2}}K=-jK=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$, is positive definite Hermitian. $C_1=\lim_{z\to 1}(z-1)G(z)=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ satisfies that $C_1+C_1^T=0$. Hence, $G(z)$ is lossless negative imaginary. We can say that $G(z)$ is lossless negative imaginary if and only if $F(z)=\frac{z-1}{1+z}[G(z)-G(-1)]=\begin{pmatrix}\frac{z^2-1}{2(z^2+1)} & -\frac{1}{2}\\ \frac{1}{2} & \frac{z^2-1}{2(z^2+1)}\end{pmatrix}$ is lossless positive real and $G(-1)=G^T(-1)$. A calculation shows that $G(-1)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, and $F(z)$ satisfies all conditions in Definition 5.

Under the assumption that $G(z)$ has no poles at $z=1$, we have the following result.

Lemma 11. *Let $G(z)$ be a square real-rational proper transfer function matrix and supposing that $G(z)$ has no poles at $z=1$. Then $G(z)$ is lossless negative imaginary if and only if*

1. $G(1)=G(1)^T$;
2. $F(z)=\frac{1+z}{1-z}(G(z)-G(1))$ *is lossless positive real.*

Proof. (Necessity) Suppose $G(z)$ is lossless negative imaginary. Then, $G(z)$ is negative imaginary. It follows from [38, Lemma 10] that $F(z)=\frac{1+z}{1-z}(G(z)-G(1))$ is positive real and $G(1)=G(1)^T$. $G(z)$ and $F(z)$ have the same poles except at $z=-1$. For any $e^{j\theta}$, $\theta\in(0,\ \pi)$, is not a pole of $G(z)$ and $F(z)$, Condition 2 of Definition 5 implies that

$$F(e^{j\theta})+F^*(e^{j\theta})=j\frac{\sin\theta}{1-\cos\theta}[G(e^{j\theta})-G^*(e^{j\theta})]=0.$$

If $G(z)$ has no poles at $z=-1$ $(\theta=\pi)$, then $F(z)$ has no poles at $z=-1$ and $F(-1)+F^T(-1)=0$. If $G(z)$ has a simple pole at $z=-1$, then $F(z)$ has also no poles at $z=-1$. Let $G(z)=\frac{C_1}{z+1}+G_1(z)$, where $C_1+C_1^T=0$ and $G_1(z)$ is analytic in $|z|>1$ and at $z=\pm 1$. Then,

$$F(z)=\frac{C_1}{1-z}+\frac{1+z}{1-z}G_1(z)-\frac{1+z}{1-z}G(1).$$

It follows that $F(-1)+F^T(-1)=\frac{C_1+C_1^T}{2}=0$. If $G(z)$ has a double pole at $z=-1$, then $F(z)$ has a simple pole at $z=-1$. Therefore, $F(e^{j\theta})+F^*(e^{j\theta})=0$ for all $\theta\in(0,\pi]$ with $e^{j\theta}$ not a pole of $F(z)$. A complex conjugate implies that $F(e^{j\theta})+F^*(e^{j\theta})=0$ for all $\theta\in[-\pi,0)$ with $e^{j\theta}$ not a pole of $F(z)$. Note that $G(z)-G(1)$ has a blocking zero at $z=1$. So $F(z)$ has no poles at $z=1$ $(\theta=0)$, and $F(1)+F^*(1)=0$ in view of the continuity of $F(z)$. Furthermore, $\theta\in[0,\pi]\cup[-\pi,0]$ is equal to $\theta\in[0,2\pi]$. Hence, we have that $F(e^{j\theta})+F^*(e^{j\theta})=0$ for all $\theta\in[0,2\pi]$ with $e^{j\theta}$ not a pole of $F(z)$. According to Definition 4, $F(z)$ is lossless positive real.

(Sufficiency) Suppose $F(z)$ is lossless positive real and $G(1)=G^T(1)$. $F(z)$ is also positive real. It follows from [38, Lemma 10] that $G(z)=\frac{1-z}{1+z}F(z)+G(1)$ is negative imaginary. Condition 2 of Definition 4 implies that $j[G(e^{j\theta})-G^*(e^{j\theta})]=\frac{\sin\theta}{1+\cos\theta}[F(e^{j\theta})+F^*(e^{j\theta})]=0$, for all $\theta\in(0,\pi)$ with $e^{j\theta}$ not a pole of $G(z)$. According to Definition 5, $G(z)$ is lossless negative imaginary. □

Remark 6. Similar to Remark 5, if $G(z)$ is symmetric, then Condition 1 in Lemmas 10 and 11 can be removed as Condition 1 in Lemmas 10 and 11 are obvious for symmetric transfer matrices.

Example 4. As an illustration of Lemma 11, consider the following transfer matrix $G(z)=\begin{pmatrix}\frac{-(z-1)^2}{(z+1)^2} & \frac{1-z}{z+1}\\ \frac{z-1}{z+1} & \frac{-(z-1)^2}{(z+1)^2}\end{pmatrix}$. $j[G(e^{j\theta}-G^*(e^{j\theta})]=0$ for all $\theta\in(0,\pi)$ with $e^{j\theta}$ not a pole of $G(z)$. $G(z)$ has double pole at $z=-1$, and $C_2=\lim_{z\to-1}(z+1)^2G(z)=\begin{pmatrix}-4 & 0\\ 0 & -4\end{pmatrix}$, $C_1=\lim_{z\to-1}(z+1)[G(z)-\frac{C_2}{(z+1)^2}]=\begin{pmatrix}4 & 2\\ -2 & 4\end{pmatrix}$, which satisfy that $C_1+C_1^*=-(C_2+C_2^*)=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}$. It can be found that $G(z)$ is lossless negative imaginary if and only if $F(z)=\frac{1-z}{1+z}[G(z)-G(1)]=\begin{pmatrix}\frac{z-1}{z(z+1)} & 1\\ -1 & \frac{z-1}{z(z+1)}\end{pmatrix}$ is lossless positive real and $G(1)=G^T(1)$. A calculation shows that $G(1)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, and $F(z)$ satisfies all conditions in Definition 4.

5. State-space conditions

In this section, we present necessary and sufficient conditions for a system to be lossless negative imaginary, in terms of minimal state-space realization conditions for both continuous-time and discrete-time systems.

5.1. Continuous-time systems

The continuous-time lossless negative imaginary lemma proposed in this subsection, which is one of the main result of the paper, extends the lossless negative imaginary lemma in [32] to the case where the transfer matrices may have poles at the origin. Theorem 3 could also be considered as a modification of in [10] applied to the lossless negative imaginary case.

Theorem 3. *Let (A, B, C, D) be a minimal state-space realization of a square real-rational proper transfer function matrix $G(s)\in\mathcal{R}^{m\times m}$, where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{m\times n}$, $D\in$*

$\mathbb{R}^{m\times m}$, and $m \leq n$. Then, $G(s)$ is lossless negative imaginary if and only if $D = D^T$, and there exists a real matrix $P = P^T \geq 0$, $P \in \mathbb{R}^{n\times n}$, such that

$$\begin{pmatrix} PA + A^TP & PB - A^TC^T \\ B^TP - AC & CB + (CB)^T \end{pmatrix} = 0. \tag{10}$$

Proof. The equivalence follows along the following sequence of equivalent reformulations.

$G(s) \sim (A, B, C, D)$ is lossless negative imaginary.

$\Leftrightarrow$ $G(\infty) = G^T(\infty)$, and $F(s) = s[G(s) - G(\infty)]$ is lossless positive real (according to Lemma 9).

$\Leftrightarrow$ $D = D^T$, and $F(s) \sim (A, B, CA, CB)$ is lossless positive real. Note that (A, B) is completely controllable and (A, CA) may be not observable. The reason is that A may be singular by allowing poles at the origin.

$\Leftrightarrow$ $D = D^T$, and there exists a real matrix $P = P^T \geq 0$, $P \in \mathbb{R}^{n\times n}$, such that

$$PA + A^TP = 0,$$
$$PB - A^TC^T = 0,$$
$$CB + (CB)^T = 0.$$

This equivalence is according to Lemma 1.

$\Leftrightarrow$ $D = D^T$, and there exists a real matrix $P = P^T \geq 0$, $P \in \mathbb{R}^{n\times n}$, such that Eq. (10) holds. □

Theorem 3 can be restated as the following corollary.

Corollary 2. *Consider the same assumptions in Theorem 3. Then, $G(s)$ is lossless negative imaginary if and only if $D = D^T$, $CB + (CB)^T = 0$, and there exists a real matrix $P = P^T \geq 0$, $P \in \mathbb{R}^{n\times n}$, such that*

$$PA + A^TP = 0, \quad \text{and} \quad PB - A^TC^T = 0.$$

Remark 7. In Theorem 3, the state-space realization (A, B, C, D) is assumed to be a minimal realization. In fact, if we remove the observability requirement of (A, C), the results in Theorem 3 also hold. Moreover, consider the generalized negative imaginary lemma in [10, Lemma 2]. Assume that (A, B) is controllable and (A, C) is not necessarily observable. The results in [10, Lemma 2] also hold by using Lemma 10 in [9] and [41, Lemma 3]. Compared to Theorem 1 in [32], Theorem 3 in this paper removes the non-singularity condition of state matrix A, that is, $\det(A) = 0$ is allowed in this paper by allowing poles at the origin, and P is allowed to be positive semidefinite. Compared to Lemma 2 in [10], the inequality in [10] is modified as equality in this paper by applying to the case where the negative imaginary transfer matrix is lossless.

5.2. Discrete-time systems

In this subsection, two discrete-time lossless negative imaginary lemmas in state-space conditions are developed to give an algebraic characterization of linear discrete-time lossless negative imaginary transfer function matrices.

Theorem 4. *Let (A, B, C, D) be a minimal state-space realization of a square real-rational proper transfer function matrix $G(z) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in$*

$\mathbb{R}^{m\times m}$, *and* $m \le n$. *Suppose* $\det(I+A) \neq 0$ *and* $\det(I-A) \neq 0$. *Then,* $G(z)$ *is lossless negative imaginary if and only if*

1. $C(I+A)^{-1}B - D = B^T(I+A^T)^{-1}C^T - D^T$;
2. *There exists a real matrix* $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, *such that*

$$Y - A^TYA = 0, \quad \text{and} \quad C = B^T(I-A^T)^{-1}Y(I+A). \tag{11}$$

Proof. The proof follows along the following sequence of equivalences.

$G(z) \sim (A, B, C, D)$ is lossless negative imaginary.

$\Leftrightarrow$ $G(-1) = G^T(-1)$, and $F(z) = \frac{z-1}{z+1}(G(z) - G(-1))$ is lossless positive real (according to Lemma 10).

$\Leftrightarrow$ $C(I+A)^{-1}B - D = B^T(I+A^T)^{-1}C^T - D^T$, and $F(z) \sim \left(\begin{array}{c|c} A & B \\ \hline C(A-I)(A+I)^{-1} & C(A+I)^{-1}B \end{array}\right)$ is lossless positive real (according to [38, Lemma 7]).

$\Leftrightarrow$ $C(I+A)^{-1}B - D = B^T(I+A^T)^{-1}C^T - D^T$, and there exists a real matrix $Y = Y^T > 0$ such that

$$Y - A^TYA = 0,$$
$$(A^T+I)^{-1}(A^T-I)C^T - A^TYB = 0,$$
$$C(A+I)^{-1}B + B^T(I+A^T)^{-1}C^T - B^TYB = 0.$$

This equivalence is according to the discrete-time lossless positive real lemma, see Lemma 4.

$\Leftrightarrow$ $C(I+A)^{-1}B - D = B^T(I+A^T)^{-1}C^T - D^T$, and there exists a real matrix $Y = Y^T > 0$ such that

$$Y - A^TYA = 0,$$
$$B^TYA(A+I)(A-I)^{-1} = C,$$
$$B^TYB - B^TY(A+I)^{-1}B - B^T(A^T+I)^{-1}YB = 0.$$

$\Leftrightarrow$ $C(I+A)^{-1}B - D = B^T(I+A^T)^{-1}C^T - D^T$, and there exists a real matrix $Y = Y^T > 0$ such that

$$Y - A^TYA = 0,$$
$$B^TY[I + (A-I)^{-1}](A+I) = C,$$
$$B^T(A^T+I)^{-1}[(A^T+I)Y(A+I) - (A^T+I)Y - Y(A+I)](A+I)^{-1}B = 0.$$

$\Leftrightarrow$ $C(I+A)^{-1}B - D = B^T(I+A^T)^{-1}C^T - D^T$, and there exists a real matrix $Y = Y^T > 0$ such that

$$Y - A^TYA = 0, \tag{12}$$
$$B^T(I-A^T)^{-1}Y(A+I) = C, \tag{13}$$
$$B^T(A^T+I)^{-1}(A^TYA - Y)(A+I)^{-1}B = 0. \tag{14}$$

$\Leftrightarrow$ $C(I+A)^{-1}B - D = B^T(I+A^T)^{-1}C^T - D^T$, and there exists a real matrix $Y = Y^T > 0$ such that

$Y - A^T Y A = 0$, and $B^T (I - A^T)^{-1} Y (A + I) = C$.

(Eq. (14) always holds.) □

The following corollary characterizes the equivalent properties of discrete-time lossless negative imaginary lemma in state-space conditions.

Corollary 3. *Suppose* $\det(I + A) \neq 0$ *and* $\det(I - A) \neq 0$. *The following two statements are equivalent:*

1. *There exists a real matrix* $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, *such that* $Y - A^T Y A = 0$ and $C = B^T (I - A^T)^{-1} Y (I + A)$.
2. *There exists a real matrix* $X = X^T > 0$, $X \in \mathbb{R}^{n\times n}$, *such that* $X - A X A^T = 0$ and $B = (I - A) X (I + A^T)^{-1} C^T$.

Proof. There exists a real matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that $Y - A^T Y A = 0$, which implies that $\det(A) \neq 0$. Then, the proof follows along the following sequence of equivalences.

There exists a real matrix $Y = Y^T > 0$ such that $Y - A^T Y A = 0$, and and $C = B^T (I - A^T)^{-1} Y (I + A)$.

$\Leftrightarrow$ There exists a real matrix $Y = Y^T > 0$ such that $Y A^{-1} - A^T Y = 0$, and $B = (I - A) Y^{-1} (I + A^T)^{-1} C^T$.

$\Leftrightarrow$ There exists a real matrix $Y = Y^T > 0$ such that $Y^{-1} (Y A^{-1} - A^T Y) Y^{-1} = 0$, and $B = (I - A) Y^{-1} (I + A^T)^{-1} C^T$.

$\Leftrightarrow$ There exists a real matrix $Y = Y^T > 0$ such that $Y^{-1} - A Y^{-1} A^T = 0$, and $B = (I - A) Y^{-1} (I + A^T)^{-1} C^T$.

$\Leftrightarrow$ There exists a real matrix $X = X^T > 0$ such that $X - A X A^T = 0$, and $B = (I - A) X (I + A^T)^{-1} C^T$ (Let $X = Y^{-1}$). □

Remark 8. The proof of Theorem 4 is completed by using the relationship between discrete-time lossless negative imaginary, lossless positive real systems, and the discrete-time lossless positive real lemma. The proof of Lemma 9 in [34] is based on the connection between discrete-time and continuous-time lossless negative imaginary transfer matrices, and the continuous-time lossless negative imaginary lemma. According to Corollary 1, it follows that Theorem 4 is equivalent to Lemma 9 in [34].

The following theorem is a new form of discrete-time lossless negative imaginary lemma:

Theorem 5. *Let* (A, B, C, D) *be a minimal state-space realization of a square real-rational proper transfer function matrix* $G(z) \in \mathcal{R}^{m\times m}$, *where* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, *and* $m \leq n$. *Suppose* $\det(I + A) \neq 0$ *and* $\det(I - A) \neq 0$. *Then,* $G(z)$ *is lossless negative imaginary if and only if*

1. $C(I - A)^{-1} B + D = B^T (I - A^T)^{-1} C^T + D^T$;
2. *There exists a real matrix* $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, *such that*

$$Y - A^T Y A = 0 \quad \text{and} \quad C = B^T (I + A^T)^{-1} Y (I - A). \tag{15}$$

Proof. The proof is similar to the proof of Theorem 4 by using the relationship between lossless positive real and lossless negative imaginary transfer matrices derived in Lemma 11, and the lossless positive real lemma introduced in Lemma 4. Details are omitted here. □

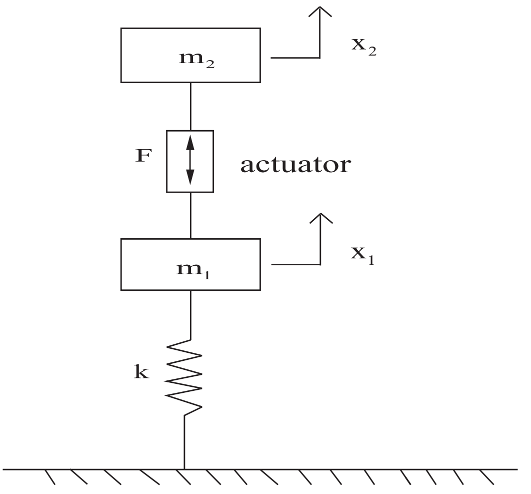

Fig. 1. Structure of undamped vibration isolation system.

Similar to Corollary 3, we have the following results, which show that Condition 2 of Theorem 4 is equivalent to Condition 2 of Corollary 4.

Corollary 4. *Suppose* $\det(I + A) \neq 0$ *and* $\det(I - A) \neq 0$. *The following two statements are equivalent.*

1. *There exists a real matrix* $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, *such that* $Y - A^T Y A = 0$ and $C = B^T (I + A^T)^{-1} Y (I - A)$.
2. *There exists a real matrix* $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, *such that* $X - AXA^T = 0$ and $B = (I + A)X(I - A^T)^{-1} C^T$.

Proof. The proof is similar to that of Corollary 3. Details are omitted here. □

6. Illustrative examples

Two examples are provided in this section. The first example provides an undamped vibration isolation system to illustrate the continuous-time lossless negative imaginary lemma. The second example illustrates the discrete-time lossless negative imaginary lemma from Single-Input Single-Output (SISO) and Multi-Input Multi-Output (MIMO) cases.

Example 5. To illustrate Theorem 3, consider an undamped vibration isolation system [42] as shown in Fig. 1. This undamped vibration isolation system is composed of two masses, an actuator and a spring. The middle mass m_1 is connected to the base through a spring k. A linear actuator is attached to the middle mass, and suspend the isolation table m_2. The actuator produces a equal and opposite force F.

Using the Newton's second law of motion and the Laplace transformation, the transfer function from $x_2 - x_1$ to F can be described by:

$$G(s) = \frac{(m_1 + m_2)s^2 + k}{m_2 s^2 (m_1 s^2 + k)}.$$

Let $m_1 = 2kg$, $m_2 = 1kg$, $k = 1N/m$. A simple choice of the transfer function could be $G(s) = \dfrac{3s^2+1}{s^2(2s^2+1)}$.

One minimal realization of $G(s)$ is as follows:

$$A = \begin{pmatrix} 0 & -0.5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1.5 & 0 & 0.5 \end{pmatrix}, \quad D = 0.$$

We can find that $D = D^T$ and $CB + (CB)^T = 0$ hold. YALMIP [43] and SeDuMi were used to find a solution of (10) as

$$P = \begin{pmatrix} 1.5 & 0 & 0.5 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0.5 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \geq 0,$$

which implies that the conditions in Theorem 3 hold. This verifies that $G(s)$ is lossless negative imaginary from state-space condition. Also, the lossless negative imaginary property of $G(s)$ can be confirmed by directly using the frequency domain conditions in Theorem 1. It can be found that all poles of $G(s)$ are purely imaginary. $G(s)$ has a double pole at $s = 0$, and $\lim_{s\to 0} s^2 G(s) = 1$. The residue matrix at $s = \frac{1}{\sqrt{2}}j$ is positive, being $\frac{\sqrt{2}}{4}$. A calculation shows that $j[G(j\omega) - G^*(j\omega)] = 0$ for all $\omega > 0$ except values of ω where $j\omega$ is a pole of $G(s)$.

Example 6.

Case 1 (SISO system)**.** To illustrate Theorems 4 and 5, we first consider a SISO system $G(z)$ with a minimal state-space realization as follows:

$$A = \begin{pmatrix} -1.2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1.2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0.92 \\ 0.4 \\ 0.92 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad D = -0.1.$$

Condition 1 in Theorems 4 and 5 are immediate. YALMIP [43] and SeDuMi were used to find a solution of the equalities in (11) as

$$Y = \begin{pmatrix} 5.0426 & -1.7756 & -2.9119 & 2.7699 \\ -1.7756 & 5.0426 & -1.7756 & -2.9119 \\ -2.9119 & -1.7756 & 5.0426 & -1.7756 \\ 2.7699 & -2.9119 & -1.7756 & 5.0426 \end{pmatrix} > 0,$$

and a solution of the equalities in Eq. (15) as

$$Y = \begin{pmatrix} 2.8232 & -0.4439 & -2.2905 & 0.6295 \\ -0.4439 & 2.8232 & -0.4439 & -2.2905 \\ -2.2905 & -0.4439 & 2.8232 & -0.4439 \\ 0.6925 & -2.2905 & -0.4439 & 2.2832 \end{pmatrix} > 0.$$

A calculation shows that the associated transfer function is given by

$$G(z) = \frac{-z^4 + 8z^3 + 2z^2 + 8z - 1}{10z^4 + 12z^3 + 20z^2 + 12z + 10},$$

which satisfies all conditions in Definition 4.

Case 2 (MIMO system)**.** We further present an MIMO example to illustrate the state-space conditions in Theorems 4 and 5. Consider a lossless negative imaginary transfer matrix without poles at $z = \pm 1$ as follows:

$$G(z) = \begin{pmatrix} \frac{-(z-1)^2}{z^2+1} & \frac{1-z^2}{2(z^2+1)} \\ \frac{z^2-1}{2(z^2+1)} & \frac{-(z-1)^2}{z^2+1} \end{pmatrix}.$$

A minimal state-space realization of $G(z)$ is as follows:

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}.$$

A calculation shows that $C(I+A)^{-1}B - D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $C(I-A)^{-1}B + D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Condition 1 in Theorems 4 and 5 hold. YALMIP [43] and SeDuMi were used to find a solution of the equalities in Eq. (11) as

$$Y = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} > 0, \tag{16}$$

Meanwhile, the matrix in Eq. (16) is also a solution of the equalities in Eq. (15).

7. Conclusions

In this paper, we present some important properties of lossless negative imaginary transfer matrices for both continuous-time and discrete-time linear time-invariant systems. Necessary and sufficient conditions are established for lossless negative imaginary transfer matrices in terms of frequency domain and state-space conditions. Although some of properties follow from known results, and are presented for the sake of completeness, new and interesting results mainly focus on the minor decomposition theory for lossless negative imaginary systems in terms of a partial-fraction expansion, and these results are important for generalising the non-proper lossless negative imaginary theory in this paper. Moreover, two new relationships between the lossless positive real and lossless negative imaginary transfer matrices have been systematically established. According to these new relationships, we develop a continuous-time generalized lossless negative imaginary lemma in terms of a minimal state-space realization, which can allow poles to be the origin, and a different version of discrete-time lossless negative imaginary lemma is derived, simultaneously. These results would lay down an important basis of the system analysis of lossless positive real and lossless negative imaginary systems with free body dynamics.

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